Convergence in discrete-time neural networks with specific performance

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We analyze convergence in discrete-time neural networks with specific performance such as decay rate and trajectory bounds in terms of componentwise absolute (exponential) stability. Simple necessary and sufficient stability and positive invariance conditions are presented, which allow us to design a convergent network with prescribed performance. Our approach is based on a decomposition of competitive-cooperative connectivity or inhibitory-excitatory interaction that abounds in neural networks, without assuming symmetry of the connection matrices. The key idea is that through the decomposition, we can always relate a competitive-cooperative network with a cooperative dynamical system. The latter possesses significant order-preserving properties that are basic to our analysis. The explicit division of connection weights into inhibitory and excitatory types offers a higher potential for relating formal neural network models to neurophysiology.

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I. INTRODUCTION

Neural networks are abstract computational models for parallel information processing. Such computational capabilities as association and optimization rely on the convergent dynamics of neural networks. In most studies, convergence in a neural network has been characterized by the monotonic decreasing of an energy function on the motion of the system, especially in the framework of symmetric connection weights [1–7]. This approach is physically simple and intuitive, but the performance of a network in the form of, e.g., the rate of convergence from an initial condition to the final state is normally difficult to assess, as energy functions are usually in an involved form of the state variables. In a practical design of a network system, the convergence rate is a critical performance that should be taken into account. Such an issue was addressed in [8–10] and sufficient results were obtained on trajectory bound estimates to allow the designer to predict the rate of convergence near the equilibria of a neural network. These results were derived by choosing certain special Lyapunov functions that may be viewed as generalized energy functions of the concerned networks, but are of a simpler form in the state variables and do not assume symmetry of the connection matrices. In this paper, we study a special type of stability, namely componentwise absolute (exponential) stability of neural networks, which characterizes in a detailed manner the convergent behavior of a neural network. We will present two necessary and sufficient stability conditions for a class of discrete-time neural networks. The results are simple and allow one to design a network with a specific decay rate and trajectory bounds.

Our approach exploits the competitive-cooperative or excitatory-inhibitory connectivity structure of a network. By competitive connection we mean the way in which a neuron's firing inhibits the firing of other neurons. Conversely, cooperative connection refers to the way in which a neuron's firing excites the firing of others. In most cases, the activation of a neuron is characterized by a sigmoid function (i.e., a continuous, bounded, and nondecreasing function). The competitive-cooperative connection pattern can thus be recognized by the sign of the weights: Positive weights are due to excitatory synapses, negative weights are due to inhibitory synapses, while a zero weight indicates no neuronal connection at all. This observation motivates a decomposition approach that eventually relates a competitive-cooperative network with an augmented cooperative system. Such a system has a significant order-preserving property that may play a key role in the analysis of the original neural network. It is well known that competitive and cooperative mechanisms abound in biological networks. The role of these mechanisms in the emergent collective dynamics in neural networks has raised great interest (e.g., [11–16]). The explicit division of connection weights into inhibitory and excitatory types offers a higher potential for relating formal neural network models to neurophysiology.

II. THE MODEL AND SPECIFICATION

Consider a class of discrete-time neural networks described by the coupled nonlinear iterative equation

$$x(k+1)=Ts[Bx(k)]+c,\quad k=0,1,\ldots, \tag{1}$$

where $x\in\mathbf{R}^n$ is the neural state vector, $T=[T_{ij}]$ is the synaptic connection matrix, $B=\mathrm{diag}[b_1,\ldots,b_n]$ with $b_i>0$ is the gain matrix, and $s(x)=[s_1(x_1),\ldots,s_n(x_n)]^T$ is a vector-valued activation function of sigmoid type (i.e., continuous, bounded, and monotonic increasing). The last term c is the constant external input vector to the networks. We assume throughout that the nonlinear function $s(x)$ satisfies

$$0\leqslant\frac{s_i(r_1)-s_i(r_2)}{r_1-r_2}\leqslant 1 \tag{2}$$

for $r_1,r_2\in\mathbf{R}^n$, $i=1,\ldots,n$. It is easy to see that the often used sigmoid functions such as $\tanh x_i$ and $\frac{1}{2}[|x_i+1|-|x_i-1|]$ possess such a property.

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Given a constant input vector c, the equilibria of system (1) are determined by

$$x_e = Ts(Bx_e) + c.$$

Since $s(x)$ is bounded and continuous, it follows immediately from Brower's fixed point theorem that there is at least one solution x_e to the above equation for every constant vector c. Further, if x_e is globally stable, then it is the unique equilibrium to which all other trajectories converge. In this case, the network (1) realizes a one-to-one mapping from the input space to the steady-state space, acting as a category classifier. Also, the global convergence is of applicable significance in neural optimization and control. One purpose of this work is to provide a global exponential stability criterion for Eq. (1).

To examine the stability of x_e, let us take the transformation $z = x - x_e$ and rewrite Eq. (1) as

$$z(k+1) = Tf[z(k)], \tag{3}$$

where $f(z) = s[B(z + x_e)] - s(Bx_e)$. It is easy to find that f belongs to the following sector nonlinear function class $\mathcal{F}$ defined by

$$\text{(i)} \quad f_i(0) = 0,$$

and

$$\text{(ii)} \quad 0 \leqslant \frac{f_i(r_1) - f_i(r_2)}{r_1 - r_2} \leqslant b_i, i = 1, \ldots, n.$$

In the following, we will study the stability of the origin for Eq. (3).

To specify the convergent performance of system (3), we consider two functions $\xi(k), \varsigma(k): [0, +\infty) \to \mathbf{R}^n$ with $\xi(k) > 0, \varsigma(k) > 0$ and

$$\lim_{k\to\infty} \xi(k) = 0 = \lim_{k\to\infty} \varsigma(k). \tag{4}$$

Then the system (3) is said to be *componentwise absolutely stable with respect to* $\gamma(k) = [\xi(k)^T \varsigma(k)^T]^T$ (CABSγ) if for every $f \in \mathcal{F}$ the solution of Eq. (3) satisfies

$$-\xi(k) \leqslant z(k) \leqslant \varsigma(k), \quad k \geqslant k_0 \geqslant 0 \tag{5}$$

whenever $-\xi(k_0) \leqslant z(k_0) \leqslant \varsigma(k_0)$. Particularly, if

$$\xi(k) = \sigma^{-k}\alpha, \quad \varsigma(k) = \sigma^{-k}\beta \tag{6}$$

for some scalar $\sigma \in (0,1)$ and two constant vectors $\alpha, \beta \in \mathbf{R}^n$ with $\alpha, \beta > 0$, then the system is called *componentwise absolutely exponentially stable* (CABES).

These concepts enable us to characterize the convergent dynamics of the network system in a more detailed manner. Notice that the above properties are insensitive to the details of the neuronal functions (i.e., valid for the whole class $\mathcal{F}$). This feature of robustness against change in the model details is of basic physical significance [1]. We also note that for linear deterministic systems, the above notions are specialized to that concerned in [17,18].

Clearly, CABSγ implies *positive invariance* of the time-dependent set

$$\Omega_{\xi,\varsigma}(k) = \{z \in \mathbf{R}^n : -\xi(k) \leqslant z \leqslant \varsigma(k)\}$$

with respect to (w.r.t.) the motion of Eq. (3) for each $k_0 \geqslant 0$. Recall that a subset of the state space is said to be positively invariant w.r.t. a dynamical system if the motion emanating from the set remains in it. Invariant sets are very special subsets of the state space since they represent regions that trap the motion of a dynamical system. For system (3), the notion of *robust positive invariance* of a set is pertinent. That is, the positive invariance of a set is valid w.r.t. the whole family of systems described by Eq. (3) with $f \in \mathcal{F}$.

The main purpose of this paper is to establish necessary and sufficient conditions for CABSγ and CABES of system (3) by using a decomposition approach to be developed in the next section.

III. COMPETITION-COOPERATION DECOMPOSITION AND COMPARISON PRINCIPLE

We split the connection matrix T into two parts:

$$T = T^+ - T^-$$

with $T_{ij}^+ = \max\{T_{ij}, 0\}$ being the excitatory weights and $T_{ij}^- = \max\{-T_{ij}, 0\}$ the inhibitory weights. Then system (3) can be rewritten as

$$z(k+1) = (T^+ - T^-) f[z(k)]. \tag{7}$$

We refer to it as a decomposition of competitive-cooperative connectivity of network (3).

Now take the symmetric transformation $y = -z$. From Eq. (7), it follows that

$$y(k+1) = T^+ g[y(k)] + T^- f[z(k)], \tag{8}$$

$$z(k+1) = T^- g[y(k)] + T^+ f[z(k)], \tag{9}$$

where $g(u) = -f(-u)$. Accordingly, we introduce the following augmented system:

$$d(k+1) = \Pi h[d(k)], \tag{10}$$

where

$$d(k) = \begin{bmatrix} p(k) \\ q(k) \end{bmatrix}, \quad h[d(k)] = \begin{bmatrix} g[p(k)] \\ f[q(k)] \end{bmatrix},$$

$$\Pi = \begin{bmatrix} T^+ & T^- \\ T^- & T^+ \end{bmatrix}.$$

Notice Π is (elementwise) non-negative. So, system (10) itself is cooperative and hence possesses the following important *order-preserving property*.

Lemma 1. Let $u(k)$ and $v(k)$ be solutions of Eq. (10). Then $u(k_0) \leqslant v(k_0)$ implies $u(k) \leqslant v(k)$ for $k \geqslant k_0 \geqslant 0$. Moreover, if $w(k)$ satisfies

$$w(k+1) \geqslant \Pi h[w(k)], \quad k = 0, 1, \ldots,$$

then $u(k_0)\leqslant w(k_0)$ implies $u(k)\leqslant v(k)$ for $k\geqslant k_0\geqslant 0$.

This property can be easily verified by an induction argument. It indicates that the states of a cooperative system will retain for all time their initial relationship, a partial ordering induced by the subset of non-negative state vectors of the state space. The result may be viewed as a discrete-time version of Kamke's comparison theorem for ordinary differential equations [19], and is also referred to as a comparison principle for iterative Eq. (10) [20].

It should be noted that for a solution $[p(k)^T q(k)^T]^T$ of system (10), neither $p(k)$ nor $q(k)$ should satisfy Eq. (3) unless $-p(k)=q(k)$ [in this case $z(k)=-p(k)=q(k)$ is a solution of system (3)]. In general, the two systems are related by the following *two-sided comparison principle*.

Lemma 2. Assume for Eqs. (3) and (10) that the initial condition $-p(k_0)\leqslant z(k_0)\leqslant q(k_0)$ holds. Then for every $k\geqslant k_0\geqslant 0$,

$$-p(k)\leqslant z(k)\leqslant q(k).$$

Hence, a solution of Eq. (10) may provide a two-sided constraint on that of Eq. (3). This enables one to deduce qualitative properties of Eq. (3) from an associated cooperative system (10). In the following, we will apply this idea to study the convergent behavior of system (3).

IV. CRITERIA FOR CABSγ AND CABES

We first present a necessary and sufficient condition for CABSγ of Eq. (3).

Theorem 1. System (3) is CABSγ if and only if for any $k_0\geqslant 0$,

$$\gamma(k+1)\geqslant \Pi\Delta\gamma(k),\ \ k\geqslant k_0, \tag{11}$$

where $\Delta=\mathrm{diag}[b_1,\ldots,b_n,b_1,\ldots,b_n]$.

Proof. We proceed to show the sufficiency of condition (11). It is clear from condition (ii) of the class $\mathcal{F}$ [or, equivalently, the sector condition (2)] that $h[\gamma(k)]\leqslant\Delta\gamma(k)$. By noticing the non-negativity of the entries of matrix Π, it follows that $\Pi h[\gamma(k)]\leqslant\Pi\Delta\gamma(k)$ for $k\geqslant k_0$. Hence, if condition (11) holds, then we have

$$\gamma(k+1)\geqslant \Pi h[\gamma(k)],\quad k\geqslant k_0. \tag{12}$$

Now consider an arbitrary $f\in\mathcal{F}$ and let $z(k)$ be the solution of the corresponding Eq. (3) with the initial value satisfying $-\xi(k_0)\leqslant z(k_0)\leqslant\varsigma(k_0)$. Taking $p(k_0)=\xi(k_0)$ and $q(k_0)=\varsigma(k_0)$, then by condition (12) and Lemma 2,

$$-p(k)\leqslant z(k)\leqslant q(k),\quad k\geqslant k_0. \tag{13}$$

Meanwhile, let $u(k_0)=[p(k_0)^T q(k_0)^T]^T=\gamma(k_0)$. From Lemma 1, one gets

$$u(k)\leqslant\gamma(k),\quad k\geqslant k_0. \tag{14}$$

This and condition (13) yield

$$-\xi(k)\leqslant z(k)\leqslant\varsigma(k),\quad k\geqslant k_0.$$

So condition (11) is sufficient for system (3) to be CABSγ.

To see the necessity of condition (11), let us suppose that system (3) is CABSγ, but condition (11) is false. Then there should exist an index $i\in\{1,\ldots,2n\}$ and a time $k_1>0$ such that

$$\xi_i(k_1+1)<T_i^+B\xi(k_1)+T_i^-B\varsigma(k_1) \tag{15}$$

when $1\leqslant i\leqslant n$, or

$$\varsigma_i(k_1+1)<T_i^-B\xi(k_1)+T_i^+B\varsigma(k_1) \tag{16}$$

when $n+1\leqslant i\leqslant 2n$. Now, consider for Eq. (3) a particular sigmoid function $f(z)$ defined by

$$f(z)=\tfrac{1}{2}[|B(z+\delta)|-|B(z-\delta)|]$$

with $\delta=\xi(k_1)+\varsigma(k_1)$. Clearly, $f(z)\in\mathcal{F}$ and

$$f[-\xi(k_1)]=-B\xi(k_1),\quad f[\varsigma(k_1)]=B\varsigma(k_1). \tag{17}$$

Then, if case (15) holds, consider the vector $z(k_1)\in\Omega_{\xi,\varsigma}(k_1)$ defined by

$$z_j(k_1)=\begin{cases}-\xi_j(k_1) & \text{if}\quad a_{ij}>0,\quad j\neq i\\ \varsigma_j(k_1) & \text{if}\quad a_{ij}\leqslant 0,\quad j\neq i\end{cases},\qquad j=1,\ldots,n.$$

From (7), (15), and (17), the ith component of $z(k_1+1)$ satisfies

$$\begin{aligned}z_i(k_1+1)&=(T_i^+-T_i^-)f[z(k_1)]\\&=T_i^+f[-\xi(k_1)]-T_i^-f[\varsigma(k_1)]\\&=-[T_i^+B\xi(k_1)+T_i^-B\varsigma(k_1)]\\&<-\xi_i(k_1+1),\end{aligned}$$

where T_i^- and T_i^+ are the ith row vectors of T^- and T^+, respectively. This implies that system (3) could not be CABSγ. A similar argument can be applied to case (16), showing the necessity of condition (11). This proves the result.

Observe that the above discussion does not depend on any particular asymptotic property of $\xi(k)$ and $\varsigma(k)$. Thus, by taking $\xi(k)$ and $\varsigma(k)$ to be two constant vectors, we obtain a special positively invariant set of system (3).

Theorem 2. For two constant vectors $\alpha,\beta\in\mathbf{R}^n$ with $\alpha>0,\beta>0$. The set $\Omega_{\alpha,\beta}=\{x\in\mathbf{R}^n:-\alpha\leqslant x\leqslant\beta\}$ is robustly positively invariant w.r.t. system (3) if and only if

$$\eta\geqslant\Pi\Delta\eta, \tag{18}$$

where $\eta=(\alpha^T\beta^T)^T$.

Now, we will give our main result concerning CABES of Eq. (3). By inserting a particular $\gamma(k)$ specified by Eqs. (6) into condition (11), one immediately obtains the following necessary and sufficient condition.

Theorem 3. System (3) is CABES if and only if there are two constant vectors $\alpha,\beta\in\mathbf{R}^n$ with $\alpha>0,\beta>0$, and a scalar $\sigma\in(0,1)$ such that

$$(\sigma I-\Pi\Delta)\eta\geqslant 0, \tag{19}$$

where $\eta=(\alpha^T\beta^T)^T$ and I is an identity matrix with appropriate dimensions. It is easily verifiable that this condition can be rewritten as

$$\mu_\infty(\Gamma^{-1}\Pi\Delta\Gamma)\leqslant\sigma<1, \tag{20}$$

where $\Gamma=\mathrm{diag}[\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n]$ and $\mu_\infty(\,)$ is the infinity matrix measure defined by $\mu_\infty(M)=\max_{1\leqslant i\leqslant l}\{m_{ii}+\Sigma_{j\neq i}|m_{ij}|\}$ for a matrix $M=[m_{ij}]_{l\times l}$.

This result relates the specific exponential decay rate and trajectory bound to system parameters and thereby provides a way to design a network with desired performance. The parameters that should be taken into account only involve the connection weights and the neuron gains, regardless of the exact features of the neurons. This merit facilitates application of the criterion in a broad range.

V. DISCUSSION AND EXAMPLE

Observe that condition (19) remains valid with $\kappa\alpha,\kappa\beta$ in place of α,β for any constant $\kappa>0$. Meanwhile, given an arbitrary initial state $z(0)$ of system (3), one can always pick a $\kappa>0$ such that $\kappa\alpha\leqslant z(0)\leqslant\kappa\beta$. Therefore, by Theorem 3,

$$\kappa\sigma^{-k}\alpha\leqslant z(k)\leqslant\kappa\sigma^{-k}\beta,\quad k\geqslant 0. \tag{21}$$

This shows that condition (19) actually gives a global exponential convergence criterion for system (3), and (21) provides a trajectory estimate.

In the symmetrical case $\alpha=\beta$, condition (19) is reduced to

$$(\sigma I-|T|B)\alpha\geqslant 0, \tag{22}$$

where $|T|=[|T_{ij}|]=T^++T^-$. It is equivalent to the matrix $I-|T|B$ being an M matrix [21], i.e., there exists a constant vector $\alpha>0$ such that

$$(I-|T|B)\alpha>0. \tag{23}$$

By the properties of the M matrix, this is also equivalent to

$$\begin{vmatrix} h_{11} & \cdots & h_{1i} \\ \vdots & \cdots & \vdots \\ h_{i1} & \cdots & h_{ii}\end{vmatrix}>0,\quad i=1,\ldots,n, \tag{24}$$

where

$$h_{ij}=\begin{cases}1-b_i|T_{ii}|, & i=j\\ -b_j|T_{ij}|, & i\neq j.\end{cases}$$

Notice that, although (19), (22), (23), and (24) are all necessary and sufficient CABES conditions, the trajectory performance that a network (3) can achieve with them may be quite different. The first two can guarantee a network to be convergent with a prescribed exponential decay rate and trajectory bounds, described, respectively, by σ and α,β. Condition (23) ensures an exponential convergence in a network along with providing an estimate of the trajectory bound, but the decay rate is not specified explicitly, while the last condition (24) only indicates exponential convergence in a network, without saying anything about decay rate and trajectory bounds.

In general, from an asymmetric exponential constraint (6), one can get a symmetric one in the following way. Rewrite condition (19) as

$$\sigma\alpha-(T^+B\alpha+T^-B\beta)\geqslant 0,$$

$$\sigma\beta-(T^-B\alpha+T^+B\beta)\geqslant 0.$$

Adding them gives

$$(\sigma I-|T|B)\rho\geqslant 0, \tag{25}$$

where $\rho=\alpha+\beta>0$ and $|T|=[|T_{ij}|]=T^++T^-$. From the above, this corresponds to a symmetric constraint on the state of Eq. (3). Therefore, the existence of a $2n$-dimensional positive vector η satisfying condition (19) is equivalent to that of an n-dimensional positive vector ρ satisfying condition (25). Evidently, an asymmetric constraint may give more accurate trajectory behavior than does a symmetric one.

Now, we give an example to illustrate the main result of the paper. Consider a two-neuron network (1) with the weight matrix

$$T=\begin{bmatrix} a & b\\ c & d\end{bmatrix},\quad a,b>0;\quad c,d<0,$$

and the gain matrix $B=\mathrm{diag}[1.5,1.3]$. The performance of the corresponding system (3) is specified by (5) and (6) with $\sigma=0.5,\alpha=(4,3)^T,\beta=(5,4)^T$. By criterion (11), with now $\Delta=\mathrm{diag}[1.5,1.3,1.5,1.3]$ and

$$\Pi=\begin{bmatrix} a & b & 0 & 0\\ 0 & 0 & -c & -d\\ 0 & 0 & a & b\\ -c & -d & 0 & 0\end{bmatrix},$$

the weights should satisfy a set of linear inequalities:

$$6a+3.9b\leqslant 2,\qquad 7.5a+5.2b\leqslant 2.5;$$
$$6c+3.9d\geqslant -2,\quad 7.5c+5.2d\geqslant -1.5.$$

Finally, we find that any weight matrix T such that

$$7.5a+5.2b\leqslant 2.5,\quad a>0,\quad b>0;$$
$$7.5c+5.2d\geqslant -1.5,\quad c<0,\quad d<0.$$

will guarantee the network to be exponentially convergent with the prescribed performance.

In summary, we have introduced concepts of CABSγ and CABES to characterize convergence in discrete-time neural networks with such specific performance as decay rate and trajectory bounds. Simple necessary and sufficient stability conditions have been obtained, which relate the system parameters to the desired convergent performance, and are therefore of practical significance in applications. We have also presented a necessary and sufficient criterion for positive invariance of a hyperrectangular set w.r.t. the networks. The results shows the efficiency of the proposed decomposition method.

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[1] J.J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554 (1982).
[2] J.J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **81**, 3088 (1984).
[3] M.A. Cohen and S. Grossberg, IEEE Trans. Syst. Man Cybern. **13**, 815 (1983).
[4] G. Carpenter, M.A. Cohen, and S. Grossberg, Science **37**, 54 (1987).
[5] L.O. Chua and L. Yang, IEEE Trans. Circuits Syst. **35**, 1257 (1988); **35**, 1273 (1988).
[6] C.M. Marcus and R.M. Westervelt, Phys. Rev. A **40**, 501 (1989).
[7] B. Kosko, *Neural Networks and Fuzzy Systems* (Prentice Hall, Englewood Cliffs, NJ, 1992).
[8] A.N. Michel, J.A. Farrell, and W. Porod, IEEE Trans. Circuits Syst. **36**, 229 (1989).
[9] A.N. Michel, J.A. Farrell, and H.F. Sun, IEEE Trans. Circuits Syst. **37**, 1356 (1990).
[10] L.T. Grujić and A.N. Michel, IEEE Trans. Circuits Syst. **38**, 1182 (1991).
[11] S. Amari and A. Arbib, *Competition and Cooperation in Neural Networks* (Springer, Berlin, 1982).
[12] A. Lapedes and R. Farber, Physica D **22**, 247 (1986).
[13] M. Lemmon and B.V.K. Vijaya Kumar, Neural Networks **2**, 193 (1989).
[14] M.W. Hirsch, Neural Networks **2**, 331 (1989).
[15] T. Chu and Z. Wang, in *Studies in Dynamics, Vibration and Control*, edited by B. Chen (Hunan University Press, Changsha, People's Republic of China, 1998), pp.150–154 (in Chinese).
[16] T. Chu and L. Huang, in *Proceedings of the Third Asian Control Conference, Shanghai 2000* (CD-ROM), TE-6-1.
[17] M. Voicu, IEEE Trans. Autom. Control **29**, 937 (1984).
[18] A. Hmamed, Automatica **33**, 1759 (1997).
[19] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations* (D. C. Heath and Company, Boston, 1965).
[20] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations* (Academic Press, New York, 1987).
[21] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences* (Academic Press, New York, 1979).